Benha University Faculty of Engineering- Shoubra Eng. Mathematics & Physics Department Preparatory Year	Final Term Exam Date: December 24, 2016 Course: Mathematics 1 – A Duration: **3** hours

- The Exam consists of one page
- Answer **All** Questions (تخلفات)
- No. of questions: 4
- Total Mark: 100

Question 1

Find **y`** from the following: **24**

(a) $y = x^3 + 3^{x^2} + 3x$ (b) $y = (x - \sec x)^{-6}$ (c) $y = \log x - \ln \sin x$

(d) $y = \tan^3 x + \cos x^3$ (e) $y^3 = x \ln x + \sin y$ (f) $y = t \sin t, \; x = t \ln t$

Question 2

(a)Find the following limits: **8**

(i) $\lim\limits_{x\to 1} \dfrac{\sqrt[3]{x} - 1}{1 - x^5}$ (ii) $\lim\limits_{x\to 0} \dfrac{\log(1+x)}{4^x - 3^x}$ (iii) $\lim\limits_{x\to 0} \dfrac{x - \tan x}{x^3 + 2x^2}$ (iv) $\lim\limits_{x\to \infty} \sqrt{\dfrac{x^8 + 2x}{x + 4x^8}}$

(b)Write the Maclurin's series of the function: $f(x) = \frac{1}{1+2x}$. **4**

(c)Sketch the curve of the function : $f(x) = x^3 - 6x^2$ **4**

(d)State and verify Rolle's theorem for, $f(x) = (x-2)^2$ in interval [1, 3]. **4**

(e)Find the integrals : $(i) \int \left(x^3 + \frac{1}{x^3}\right) dx$ $(ii) \int (\sin x - \cos x)^2 \, dx$ **6**

Question 3

(a)Prove that : $4^n - 1$ is divisible by 3 for any integer $n \geq 1$. **6**

(b) *Find the sum* : $\sum_{r=1}^{n} (2r-1)(2r+1)$ **6**

(c)Determine the coefficient of x^{20} in the expansion :
$(1 + x + x^2 + x^3 + x^4)(1 - x)^{-4}$ **6**

(d)Find the cubic root of the complex number : $z = 1 + i$. **6**

Question 4

(a)Resolve the fraction : $\frac{x^4 - x^3 - x - 1}{x^3 - x^2}$ into its partial fractions. **6**

(b)Write the first four terms in the expansion : $\sqrt{4 + 2x}$. **6**

(c)Use the Gauss-Jordan algorithm to solve the linear system :
$-5x_1 - 2x_2 + 2x_3 = 14, \; 3x_1 + x_2 - x_3 = -8, \; 2x_1 + 2x_2 - x_3 = -3$ **7**

(d)Show that $x = 1 + 3i$ is a zero to the polynomial
$f(x) = x^4 - 2x^3 + 9x^2 + 2x - 10$ and then solve the equation f(x) = 0. **7**

Good Luck *Dr. Mohamed Eid* *Dr. Fathi Abdsallam*

Model Answer

Answer of Question 1

(a) $y` = 3x^2 + 3^{x^2}. \ln 3 . 2x + 3$

(b) $y` = -6(x - \sec x)^{-7}(1 - \sec x . \tan x)$

(c) $y` = \frac{1}{\ln 10} . \frac{1}{x} - \frac{\cos x}{\sin x}$

(d) $y` = 3\tan^2 x. \sec^2 x - \sin x^3 . 3x^2$

(e) $3y^2. y` = 1 + \ln x + \cos y . y`$

(f) $y` = \frac{t.\cos t + \sin t}{1 + \ln t}$

--*24-Marks*

Answer of Question 2

(a)(i) $\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{1 - x^5} = \frac{0}{0} = -\frac{1}{15}$

(ii) $\lim_{x \to 0} \frac{\log(1 + x)}{4^x - 3^x} = \frac{0}{0} = \frac{1/\ln 10}{\ln(4/3)}$

(iii) $\lim_{x \to 0} \frac{x - \tan x}{x^3 + 2x^2} = \frac{0}{0} = \lim_{x \to 0} \frac{1 - \sec^2 x}{3x^2 + 4x} = \frac{0}{0} = \lim_{x \to 0} \frac{-2\sec^2 x . \tan x}{6x + 4} = 0$

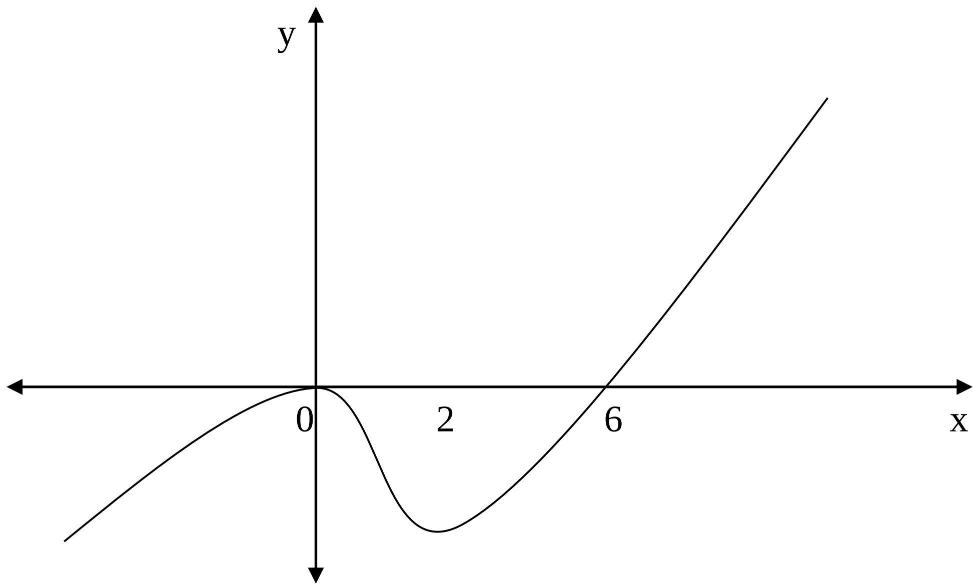
(iv) $\lim_{x \to \infty} \sqrt{\frac{x^8 + 2x}{x + 4x^8}} = \frac{\infty}{\infty} = \frac{1}{2}$

--*8-Marks*

(b) $f(x) = \frac{1}{1+2x} = 1 - 2x + 4x^2 - \cdots$

--*4-Marks*

(c)



--*4-Marks* (d)We see that (x) , its derivative $f`(x) = 2(x-2)$ are continuous in the given interval [1, 3] and f(1) = f(3) =1.

Then $f`(c) = 2(c-2) = 0$. Hence c = 2.

--*4-Marks*

$(e)(i)\int\left(x^3+\frac{1}{x^3}\right)dx = \frac{1}{4}x^4 - \frac{1}{2}x^{-2} + c$

$(ii)\int(\sin x-\cos x)^2\,dx = \int(1-2\sin x\cos x)dx = \int(1-\sin 2x)dx = x+\frac{1}{2}\cos 2x + c$

--*6-Marks*

Dr. Mohamed Eid

<u>Answer of Question 3</u>

(a) Prove that $\mathbf{4^n - 1}$ divisible by 3 for any integer $\boldsymbol{n \geq 1}$.

Proof:

STEP 1: For n=1 $\boldsymbol{P(1)}$ is true, since $\mathbf{4^1 - 1}$ divisible by 3.

STEP 2: Suppose $\boldsymbol{P(k)}$ is true for some $\boldsymbol{n \geq k \geq 1}$, that is $\boldsymbol{4^k - 1}$ divisible by 3

STEP 3:Prove that $\boldsymbol{P(k+1)}$ is true for $\boldsymbol{n \geq k+1}$, that is $\boldsymbol{4^{k+1} - 1}$ divisible by 3

We have $\boldsymbol{4^{k+1} - 1 = 4.4^k - 1 = (3+1)4^k - 1 = (3.4^k) + (4^k - 1)}$

Which show that $\boldsymbol{4^{k+1} - 1}$ divisible by 3 then $\boldsymbol{P(n)}$ true

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(b)Find the sum $\sum_{r=1}^{n}(2r-1)(2r+1)$

<u>Answer</u>

$\boldsymbol{u_r = (2r-1)(2r+1)}$

$$\boldsymbol{= \frac{(2r-1)(2r+1)(2r+3)}{6} - \frac{(2r-3)(2r-1)(2r+1)}{6} = f(r+1) - f(r)}$$

$$\boldsymbol{S_n = \frac{(2n-1)(2n+1)(2n+3)}{6} + \frac{1}{2}}$$

(c)Find coefficient $\boldsymbol{x^{20}}$ in the expansion $\boldsymbol{(1+x+x^2+x^3+x^4)(1-x)^{-4}}$.

Answer

$$(1+x+x^2+x^3+x^4) = \left(\frac{1-x^5}{1-x}\right)(1-x)^{-4} = \left(1-x^5\right)(1-x)^{-5}$$

$$=\left(1-x^5\right)\left(\sum_{r=0}^{\infty} C_r^{5+r-1}\, x^r\right)=\left(1-x^5\right)\left(\sum_{r=0}^{\infty} C_r^{r+4}\, x^r\right)$$

Coefficient is $\boldsymbol{C_{20}^{24}-C_{15}^{19}}$

(d) Find the cubic roots for the complex number $\boldsymbol{z=1+i}$

Solution:

First we put the given complex number $\boldsymbol{z=1+i}$ in the polar form

$\therefore \mathbf{1+i=\sqrt{2}\ (\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})}$ Then

$$(1+i)^{1/3}=\mathbf{(\sqrt{2})^{1/3}\ [\cos(\frac{\pi/4+2k\pi}{3})+i\sin(\frac{\pi/4+2k\pi}{3})],}$$

$\boldsymbol{k=0,1,2}$

at $\boldsymbol{k=0}$ we determine the first root

$$\mathbf{z_0=(\sqrt{2})^{1/3}\ [\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}]}$$

at $k=1$ we determine the second root

$$z_1=(\sqrt{2})^{1/3}\ [\cos(\frac{\pi/4+2\pi}{3})+i\sin(\frac{\pi/4+2\pi}{3}) \quad =(\sqrt{2})^{1/3}\ [\cos\frac{9\pi}{12}+i\sin\frac{9\pi}{12}]$$

at $k=2$ we determine the third root $z_2=(\sqrt{2})^{1/3}\ [\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}]$

Answer of Question 4

(a)Resolve the fraction $\boldsymbol{\frac{x^4-x^3-x-1}{x^3-x^2}}$ into its partial fractions.

Solution:

The fraction is an improper fraction. By division

$$\boldsymbol{\frac{(x^4-x^3-x-1)}{x^3-x^2}=x-\frac{x+1}{x^3-x^2}=x-\frac{x+1}{x^2(x-1)}} \qquad \mathbf{(1)}$$

$$\boldsymbol{\frac{x+1}{x^2(x-1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{(x-1)}} \qquad \mathbf{(2)}$$

$$\boldsymbol{\frac{x+1}{x^2(x-1)}=\frac{Ax(x-1)+B(x-1)+Cx^2}{x^2(x-1)}}$$

$$\boldsymbol{\therefore x+1=Ax(x-1)+B(x-1)+Cx^2} \qquad \mathbf{(3)}$$

For $\boldsymbol{x=0\Rightarrow 1=-B\Rightarrow \boxed{B=-1}}$

For $x=1 \Rightarrow 2=C \Rightarrow \boxed{C=2}$

For $x=2 \Rightarrow 3=2A+B+4C \Rightarrow \boxed{A=-2}$

$$\frac{x+1}{x^2(x-1)}=\frac{-2}{x}+\frac{-1}{x^2}+\frac{2}{(x-1)}$$

(b)Use the Gauss-Jordan algorithm to solve each of the following systems of linear equations.

$-5x_1-2x_2+2x_3=14$

$3x_1+x_2-x_3=-8$

$2x_1+2x_2-x_3=-3$

Solution

The augmented matrix of the system is $\left(\begin{array}{ccc|c} -5 & -2 & 2 & 14 \\ 3 & 1 & -1 & -8 \\ 2 & 2 & -1 & -3 \end{array}\right)$

Applying Gauss-Jordan algorithm to this matrix yields

$$\left(\begin{array}{ccc|c} 1 & 2/5 & -2/5 & -14/5 \\ 0 & -1/5 & 1/5 & 2/5 \\ 0 & 6/5 & -1/5 & 13/5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2/5 & -2/5 & -14/5 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

$\sim \left(\begin{array}{ccc|c} 1 & 2/5 & 0 & -4/5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right)$ the system has unique solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$

(c)Show that the value $x=1+3i$ is a zero to the polynomial

$f(x)=x^4-2x^3+9x^2+2x-10$ hence solve the equation $f(x)=0$

Solution

David the given polynomial by the linear expression $(x-1-3i)$ and we found that the remainder $R=f(x-1-3i)=0$ this means that the value $1+3i$ is a zero of $f(x)$ and $1+3i$ is a root for $f(x)=0$, $1-3i$ is another root and

$f(x)=(x-1-3i)(x-1+3i)Q(x) \qquad =(x^2-2x+10)Q(x)$

The polynomial $Q(x)$ is obtained by synthetic division as follows

1	1	-2	9	2	-10
2		2	-10		
-10			0	0	
				-2	10
	1	0	-1	0	0

Then $f(x)=(x^2-2x+10)(x^2-1)=(x^2-2x+10)(x-1)(x+1)$

The roots are $1\pm 3i, 1, -1$

(d)Write the first four terms in the expansion $\sqrt{\mathbf{4+2x}}$

Answer

$$\sqrt{4+2x}=2(1+\frac{x}{2})^{\frac{1}{2}}=2\left[1+\frac{1}{2}(\frac{x}{2})+\frac{1}{2!}(\frac{1}{2})(\frac{1}{2}-1)(\frac{x}{2})^2+\frac{1}{3!}(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{x}{2})^3+...\right]$$

$$=2\left[1+\frac{x}{4}-\frac{x^2}{32}+\frac{x^3}{128}+...\right]$$

Dr. Fathi Abdsallam